$$\lambda_1^2 = \frac{\omega_0'}{T} \frac{\partial P}{\partial \tau^*} < 0, \qquad \lambda_2(0) = \frac{1}{2} \left\langle \left(\frac{\partial f}{\partial a}\right)_0 + \left(\frac{\partial F}{\partial \psi}\right)_0 \right\rangle < 0$$

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THE MOVING ANGULAR VELOCITY HODOGRAPH IN HESS' SOLUTION OF THE PROBLEM OF MOTION OF A BODY WITH A FIXED POINT

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In contrast to other cases of integrable equations of motion of a body with a fixed point, determination of the components of the angular velocity of such a body in moving coordinates in Hess' solution [1] is not reducible to quadratures; it reduces to a Riccati differential equation, which complicates investigation considerably.

The case of integrability pointed out by Hess has been investigated by many authors, largely by analytical methods [2–7]. A geometric interpretation of the motion of a body in this solution was given by Zhukovskii [8], who used an intermediate moving coordinate system.

The present paper contains a direct interpretation (i.e. one which does not involve intermediate coordinate systems) based on Kharlamov's dynamic [9 and 10] and kinematic [11 and 12] equations.

By forgoing the principal coordinate axes usually employed in this problem in favor of Kharlamov's special axes, we are able to obtain a relatively simple equation for the moving angular velocity hodograph.

1. The motion of a body in the case in question can be conveniently studied in the special coordinate system proposed by Kharlamov. The Hess conditions and the fourth integral here become [11] $a_1 = a_2 = a^*, \quad x = 0$

Let us rotate the coordinate axes about the first axis in such a way that $b_2 = 0$ and write out the equations and integrals of motion,

$$dy / dt = b_1 yz - \nu_2 \Gamma, \qquad dz / dt = - b_1 y^2 + \nu_1 \Gamma$$

$$d\nu / dt = a^* (z\nu_1 - y\nu_2), \quad d\nu_1 / dt = by\nu_2 - az\nu, \quad d\nu_2 / dt = y (a\nu - b\nu_1) \tag{1.1}$$

$$\tfrac{1}{2} a^* (y^2 + z^2) - \nu\Gamma = E, \quad y\nu_1 + z\nu_2 = k, \quad \nu^2 + \nu_1^2 + \nu_2^2 = 1$$

The components of the angular velocity vector can be written as

$$\omega_1 = b_1 y, \ \omega_2 = a^* y, \ \omega_3 = a^* z \tag{1.2}$$

If $b_1 = 0$ we have the Lagrange case. We therefore assume from now on that $b_1 \neq 0$. Let $b_1 > 0$. Introducing the dimensionless variables y', z', τ, ω_i,

$$y = \sqrt{\Gamma / b_1}\, y', \quad z = \sqrt{\Gamma / b_1}\, z', \quad t = \tau / \sqrt{\Gamma b_1}, \quad \omega_i = a^* \sqrt{\Gamma / b_1}\, \omega_i'$$

and the parameters

$$c = 2b_1 / a^*, \quad h = E / \Gamma, \quad k' = k \sqrt{b_1 / \Gamma} \tag{1.3}$$

we find from (1.1), (1.2) that

$$y^{\cdot} = yz - \nu_2, \ z^{\cdot} = - y^2 + \nu_1 \tag{1.4}$$

$$\nu^{\cdot} = 2 (z\nu_1 - y\nu_2) / c, \ \nu_1^{\cdot} = y\nu_2 - 2z\nu / c, \ \nu_2^{\cdot} = y (2\nu / c - \nu_1) \tag{1.5}$$

$$y^2 + z^2 - c\nu = ch, \ y\nu_1 + z\nu_2 = k, \ \nu^2 + \nu_1^2 + \nu_2^2 = 1 \tag{1.6}$$

$$\omega_1 = \tfrac{1}{2} cy, \ \omega_2 = y, \ \omega_3 = z \tag{1.7}$$

For convenient notation we have omitted the primes in relations (1.4)–(1.7). The dot denotes differentiation with respect to the dimensionless time τ.

From Eqs. (1.4) with allowance for (1.6) we obtain

$$\tfrac{1}{2} (y^2 + z^2)^{\cdot} = \sqrt{(y^2 + z^2) \{1 - [(y^2 + z^2) / c - h]^2\} - k^2}$$

$$y\dot{z} - z\dot{y} = - y (y^2 + z^2) + k$$

Next, we introduce the polar coordinates $y = \rho \cos \varphi$, $z = \rho \sin \varphi$. We then obtain a system of two differential equations for determining ρ, φ,

$$\rho\rho^{\cdot} = \sqrt{f(\rho)}, \quad \rho^2 \varphi^{\cdot} = - \rho^3 \cos \varphi + k \left(f(\rho) = \rho^2 \left[1 - \left(\frac{\rho^2}{c} - h \right)^2 \right] - k^2 \right) \tag{1.8}$$

The dependence of ρ on φ is given by the equation

$$\frac{d\varphi}{d\rho} = \frac{\rho^2 \cos \varphi - k}{\rho \sqrt{f(\rho)}} \tag{1.9}$$

2. Let us determine the domain of variation of the parameters c, h, k. To find the range of variation of c we express it in terms of the components of the inertia tensor. Setting $A_{11} >$, we have $A_{22} > A_{33}$, $c = 2 \sqrt{(A_{22} - A_{33}) / A_{11}}$, $\quad 0 < c \leqslant 2$

The first equation of (1.8) has real solutions only for $f(\rho) \geqslant 0$. This imposes a restriction on one of the parameters h, k.

Let us leave k as the independent parameter. The range of variation of h depends on k.

In order for $f(\rho) \geqslant 0$ it is necessary that $f(\rho_0) \geqslant 0$, where ρ_0 is the largest root of the equation $f'(\rho) = 0$. Hence, $h > h_* > -1$; in this condition h_* is the largest root of the equation defining h, $2c[9h - h^3 + (h^2+3)\sqrt{h^2+3}] - 27k^2 = 0$

As $|k|$ decreases to zero, h_* decreases to -1.

3. From (1.7) we see that the moving hodograph lies in the plane $\omega_1 = {}^1/_2\, c\omega_2$, and that the projection of the moving hodograph on the plane $\omega_1 = 0$ is the curve s defined by system (1.8) or by the equivalent equation (1.9).

Let us investigate this curve. The projection of the moving hodograph lies in the ring G: $\rho_1 \leqslant \rho \leqslant \rho_2$, where ρ_1, ρ_2 are the positive roots of the equation $f(\rho) = 0$. The existence of these roots is proved by the fact that

$$f(0) < 0, \quad f(\rho_0) \geqslant 0, \quad f(\infty) < 0$$

It is essential that $\rho_1 \geqslant 0$, and that ρ can vanish only for $k = 0$.

The shape of the curve s depends essentially on the number of singular points of Eq. (1.9). We denote the singular points as follows:

$$M_1(\rho_1, \varphi_1),\ M_{-1}(\rho_1, -\varphi_1),\ M_2(\rho_2, \varphi_2),\ M_{-2}(\rho_2, -\varphi_2)$$

$$\varphi_1 = \arccos(k/\rho_1^3), \quad \varphi_2 = \arccos(k/\rho_2^3)$$

The number of singular points of the equation under consideration is not larger than four. Let us stipulate that $h_1 = (k^{2/3} - c\sqrt{1-k^{4/3}})/c, \qquad h_2 = (k^{2/3} + c\sqrt{1-k^{4/3}})/c$

We distinguish five cases according to the number of singular points.

1. Equation (1.9) has four singular points when $\rho_1^3 > |k|$. This is the case for all $h > h_*$ if $|k| \geqslant 1$, and for $h > h_2$ if $|k| < 1$.
2. Equation (1.9) has three singular points if $\rho_1^3 = |k|$. Here $|k| < 1$, $h = h_2$.
3. Equation (1.9) has two singular points if the interval (ρ_1, ρ_2) contains a value ρ_* such that $\rho_*^3 = |k|$. Here $|k| < 1$, $h_1 < h < h_2$.
4. Equation (1.9) has one singular point if $\rho_2^3 = |k|$; here $|k| < 1$, $h = h_1$.
5. Equation (1.9) has no singular points if $\rho_2^3 < |k|$; here $h_* < h < h_1$.

In order to investigate the curve s let us construct the direction field for Eq. (1.9):

$$\operatorname{tg}\vartheta = \rho\frac{d\varphi}{d\rho}, \quad \text{or} \quad \operatorname{tg}\vartheta = \frac{\rho^3\cos\varphi - k}{\sqrt{f(\rho)}} \tag{3.1}$$

Here ϑ is the angle between the positive direction of the coordinate ρ and the tangent to the curve at this point; it is measured counterclockwise.

Equation (3.1) defines two directions at each point of the domain, since $\sqrt{f(\rho)}$ has two values differing in their sign at each point. These directions are symmetric with respect to the ray constructed from the origin to the given point. It is therefore sufficient to investigate the direction field, assuming that $\sqrt{f(\rho)}$ in expression (3.1) is a positive quantity.

When the coordinate $\rho = \rho_1, \rho_2$ we have $tg\vartheta = \infty$, i.e. the curve s tangent to the inner and outer boundaries of the ring G, respectively. The domain G_1 in which $\operatorname{tg}\vartheta > 0$ is separated from the domain G_2 in which $\operatorname{tg}\vartheta < 0$ by the curve L: $\rho^3\cos\varphi - k = 0$ along which $\operatorname{tg}\vartheta = 0$. The curve L passes through the singular points of the equation under investigation.

Let us consider the variation of $\operatorname{tg}\vartheta$ as the variable point moves along a radius from one boundary of the ring G to the other. Along a radius which does not intersect the line L the absolute value $|\operatorname{tg}\vartheta|$ diminishes from infinity to some value and then increases again without limit.

If the radius intersects the line L, then $\operatorname{tg}\vartheta$ varies from infinity to zero, which it reaches at the point of intersection of the radius with the line L. Here $\operatorname{tg}\vartheta$ changes sign, and if the point reaches the boundary of the domain G, then $\operatorname{tg}\vartheta$ once again becomes infinite.

If $k<0$, then the direction field (and the curve L) are symmetric to the direction field for $k>0$ with respect to the z-axis. For this reason we shall first consider the case $k>0$.

From the first equation of (1.8) we find that ρ is an elliptic function of the time τ with the period

$$T=2\int_{\rho_1}^{\rho_2}\frac{\rho\,d\rho}{\sqrt{f(\rho)}}$$

The polar angle φ can be found from the second equation of (1.8). If the initial point is chosen in the domain G_1, then φ begins to increase with increasing time; if it is chosen in G_2, then φ begins to decrease with increasing time. Hence for each of the cases 1–5 we must consider two variants corresponding to the two initial positions of the initial point.

1) the initial point lies in G_1 ($\varphi=\pi$);
2) the initial point lies in G_2 ($\varphi=0$).

4. To continue our investigation we introduce the symbols $S_k^1, S_k^2, s_k^1, s_k^2$, which are the points where the curves s^1, s^2 are tangent to the circles $\rho=\rho_2$, $\rho=\rho_1$.

Let α_k^1,α_k^2 be the values of the polar angle φ for which s^1, s^2 intersect the line L.

Here and below the superscript denotes the number of the domain G_p in which the chosen initial point lies.

Let us establish two important properties of the curve s which follow from the singularity of the direction field of Eq. (1.9).

A) The arcs $S_i^1s_i^1$, $S_i^2s_i^2$ of the curve s cannot intersect the arcs $S_j^1s_j^1$ and $S_j^2s_j^2$ for any i, j, and can intersect the arcs $s_j^1S_{j+1}^1$, $s_j^2S_{j+1}^2$ in the symmetric direction.

B) For all i we have $\alpha_{2i}>\alpha_{2i-1}^1$ and $\alpha_{2i}^2<\alpha_{2i+1}^2$.

The arcs $S_j^1s_j^1$, $S_j^2s_j^2$ are obtainable as the solutions of system (1.8) for the time interval $(0, {}^1/_2T)$, when we take S_j^1, S_j^2 as our initial points. All the conditions of existence and uniqueness of the solution of system (1.8) are fulfilled in this case, i.e. a single integral curve (or, which is the same thing, a single arc $S_j^1s_j^1$, $S_j^2s_j^2$) passes through every point of the domain G. This proves the first half of the statement (A).

If we change the sign of the radical in the first equation of (1.8), then the solutions of the resulting system (1.8) in the interval $(0, {}^1/_2T)$ are the arcs $s_j^1S_{j+1}^1$, $s_j^2S_{j+1}^2$, provided we take s_j^1, s_j^2 as our initial points. This and the preceding consideration imply the validity of the second part of the statement (A).

Property (B) is self-evident.

Let us note a corollary of Property (A).

C) If we associate the points $S_k^1, s_k^1, S_k^2, s_k^2$ with the values of their arc coordinates whose origins are some points of the circles $\rho=\rho_1$, $\rho=\rho_2$ (we consider the counterclockwise direction positive), then Property (A) implies that the sequences S_k^1,

s_k^1, S_k^2, s_k^2 are strictly monotonic.

Now let us investigate the cases 1–5.

1a. Let us take a point S_0^1 lying in the domain G_1 on the circle $\rho = \rho_2$ which is not a singular point of Eq. (1.9), and study the motion of the variable point M of the curve s.

At the initial instant the curve s^1 is tangent to the circle $\rho = \rho_2$. With increasing time ρ begins to decrease and φ begins to increase. The point M emerges from S_0^1 and reaches the point s_0^1 after the time $^1/_2T$; at the same time the curve s^1 is tangent to the circle $\rho = \rho_1$ (the curve s_1 has an inflection point in the interval (ρ_1, ρ_2)). At this instant the radical in the first equation of (1.8) changes sign, after which ρ increases. The curve s^1 again passes through the inflection point, touches the outer circle at the point S_1^1 for $\tau = T$, etc. The piece of the curve s which contains inflection points is of the type F_1. The piece of the curve s^1 which does not intersect the line L is of the type F_1.

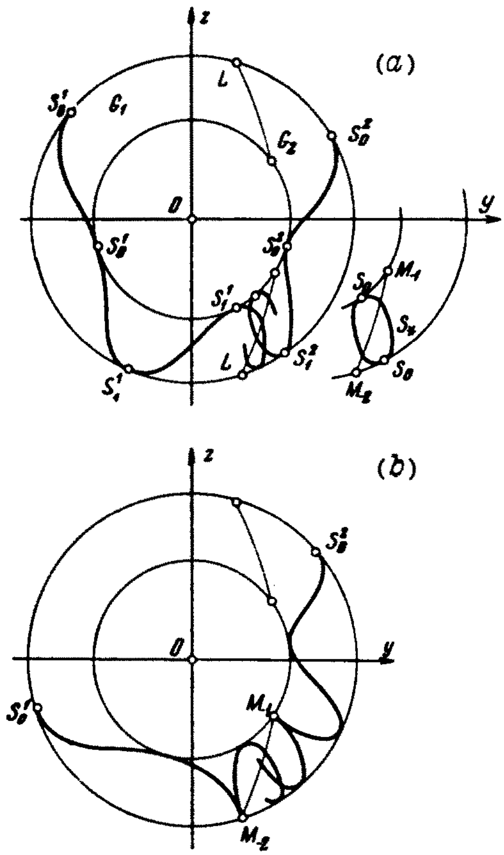

Fig. 1

Let S_m^1 be the first of the points S_i^1 lying in the domain G_2. On intersecting the line L the curve s^1 passes from the domain G_1 into the domain G_2, whereupon the angle φ begins to decrease. The point M passes through the point S_m^1, and for some value of τ leaves the domain G_2 (the curve s^1 intersects the line L a second time). By virtue of property (B) $\alpha_0^1 > \alpha_1^1$, we infer from Corollary (C) that $s_m^1 > s_{m-1}^1$, so that the point M with increasing time intersects the arc $s_{m-1}^1 S_m^1$ and reaches the point s_m^1. The curve s^1 then again intersects the line L and the picture is repeated. The piece of the curve s containing self-intersection points but no inflection points is of the type F_2. The piece of the curve s^1 which intersects the line L is of the type F_2.

It is clear that the curve s^2 which results if we take some point S_0^2 of the domain G_2 as our initial point is of the same shape as the curve s^1.

If s_e^2 is the first of the points s_i^2 which lies in the domain G_1, then the points S_i^1, S_j^2 do not leave the arc $S_m^1 S_e^2$ of the circle $\rho = \rho_2$, and the points s_i^1, s_j^2 do not leave the arcs $s_m^1 s_e^2$ of the circle $\rho = \rho_1$ for $i > m$, $j > e$. As $\tau \to \infty$, the points S_i^1, S_i^2 tend to S_0 and the points s_i^1, s_i^2 to s_0; the trajectories of the variables point M of the curves s^1, s^2 approach without limit the limiting closed trajectory S_* which passes through the points s_0, S_0. The trajectory S_* is the limiting cycle for the curves s^1, s^2.

The trajectory S_* is unique for all initial positions of the point for the chosen parameter values, i. e. for tne chosen domains G, G_1, G_2. This is proved by the fact that on leaving the point in G_2 fixed and taking as out initial point any point from the domain G_1, we find that the trajectory S_* is the limiting cycle of the curves s^1 when any points of the domain G_1 are taken as the initial points. Similarly, holding fixed some initial point in G_1, we find that S_* is the only limiting cycle for the curves s^2 for any positions of the initial point in G_2. Fig. 1a shows the curves s^1, s^2. The limiting cycle S_* is shown separately in Fig. 1a (*).

1b. Let us consider the existence of the curve s^1 which passes through the singular point M_{-2}. Let us take the latter as our initial point; since in this position the velocity of the point M of. the curve s^1 is equal to zero, the transverse and radial components of the acceleration at this point are $w_\varphi = 0$, $w_\rho = F'(\rho_2) / 2\rho_2^2$

From this we see that the point M emerges from the point M_{-2}; at this time the curve s^1 is tangent to the radius.

The subsequent behavior of the curve does not differ from that described earlier for the case where the curve s intersects the line L. The curve s^1 is of the type F_2; its limiting cycle is S_*.

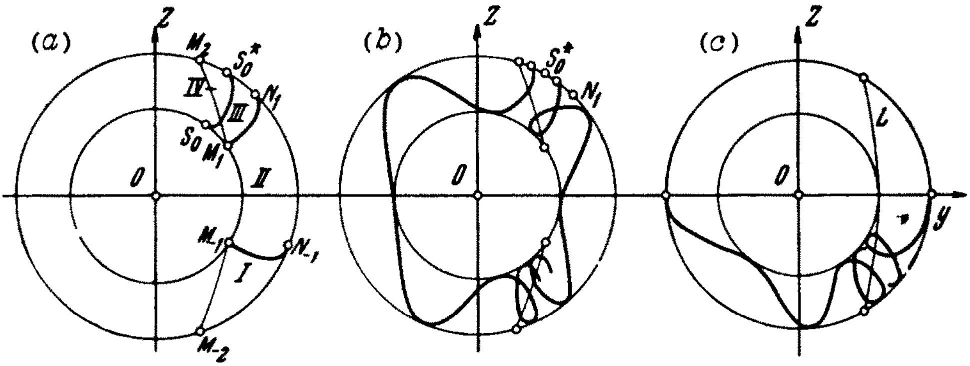

Fig. 2

If we replace τ by $-\tau$ in the second equation of (1. 8), the resulting system (1. 8) defines the curve s^1 as τ varies from 0 to $-\infty$. Here the angle φ decreases, the point M emerges from the point M_{-2}, and the curve s^1 is tangent to the radius. The subsequent behavior of the curve s^1 is as described in Sect. 1a for the case where the curve s^1 does not intersect the line L, i. e. where the curve s^1 is of the type F_1.

Now, taking some point of this curve situated on the circle $\rho = \rho_2$ as our initial point, we obtain the curve $\sigma_{-2}{}^1$ which passes through a singular point of Eq. (1. 9) and has S_* as its limiting cycle; the point M_{-2} is a cusp of the curve $\sigma_{-2}{}^1$. In the same way we find that the domain G_2 contains a trajectory $\sigma_{-1}{}^2$ which passes through the singular point M_{-1}. The shape of the curve s in this case is shown in Fig. 1b.

Let us consider the shapes of the curve s^2 for various initial points in the domain G_2. To do this we extend the curves s^1, s^2 for negative values of the time. As $\tau \to -\infty$ the trajectory of the variable point M of the given curves $s^{\circ 1}$, $s^{\circ 2}$ approaches without limit

*) All of the letters S at the inner circles in Figs. 1–4 are small; all of the letters S at the outer circles are capitals. The symbol S^* in the figures appears as S^* in the text.

the unique closed trajectory S_*° passing through the points s_0°, S_0°. We note that the trajectory S_*° is symmetric to the trajectory S_* with respect to the y-axis.

It is easy to show that the domain G_2 contains two trajectories $\sigma_1^2, \sigma_{-1}^2$ passing through the singular points M_1, M_{-1}, respectively.

Let N_1, N_{-1} be the points of the curves σ_1^2, σ_{-1}^2 lying on the circle $\rho = \rho_*$ which directly precede M_1, M_{-1}.

The points M_1, M_{-1}, M_2, M_{-2}, N_1, N_{-1}, s_0°, S_0° divide G_2 into four domains (Fig. 2a); the shape of the curve s^2 depends on the domain in which the initial point is chosen.

The segments of the curve s^2 lying in domains I, II, IV are of the type F_2; that lying in III is of the type F_1.

When points in domains II or I are chosen as the initial points, the curves s^2 can be found from the line shown in Fig. 1a if as our initial point we take S_0^2, S_1^2, respectively.

Fig. 2b shows the curves s^2 when the initial points lie in domains III and IV. We note that if the initial point lies in domain IV, then the curve s^2 leaves the domain G_2 and passes through the entire domain G_1.

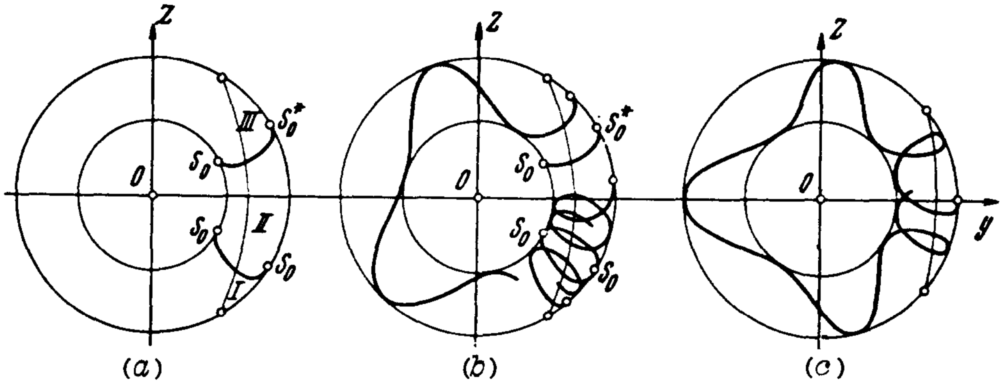

Fig. 3

In precisely similar fashion the domain G_1 breaks down into four subdomains according to the shape of the curve s^1.

Thus, the curve s in the case just considered has a unique limiting cycle S_* for any initial point. If the initial point lies on the trajectories S_*, S_*°, then the curve s is closed. Each singular point of Eq. (1. 9) is associated with a single trajectory which passes through this point. The singulat point is the cusp of each curve. In our case there are four trajectories which have a cusp.

2. The curve s for this case (Fig. 2b) can be obtained by the same reasoning as above. Division of G_1, G_2 into domains characterized by a specific shape of the curve s can be effected as in Case 1.

As in the above case, there exist two closed trajectories obtainable by appropriate choice of the initial point. For any other positions of the initial point the curves s^1, s^2 have S_* as their only limiting cycle.

In contrast to the previous case, the present choice of parameters gives rise to three trajectories with cusps.

3. Let us take the point S_0^2 ($\varphi = 0$) of the domain G_2 as our initial point. The coordinates ρ and φ decrease with increasing time. As soon as the curve s^2 intersects the

line L the angle φ begins to increase; ρ begins to increase at the instant $\tau = {}^1/_2 T$ when the line s^2 touches the circle $\rho = \rho_1$. When the trajectory s^2 intersects the line L the second time, the angle φ again begins to diminish; at the instant $\tau = T$ the curve s^2 touches the outer circle at the point S_1^2.

We have three possible variants:

(a) $S^2_1 < S_0^2$, (b) $S_1^2 > S_0^2$, (c) $S_1^2 = S_0^2$

Let us investigate the curves s^2 for each of these variants.

a) Considering the curve s^2 jointly with s^1, we find that the trajectory of the variable point M of the curves s^1, s^2 approaches the closed curve S_* without limit as $\tau \to \infty$. As in Case 1 there exists a closed curve S_*° symmetric to S_* with respect to the y-axis.

Let us investigate the curve s^2 for various positions of the initial points in the domain G_2. The arcs $s_0 S_0$, $s_0^\circ S_0^\circ$ divide G_2 into three domains (Fig. 3a). For domaind I and III we have $S_{i+1}^2 > S_i^2$; in domain II, by hypothesis, $S_{i+1}^2 < S_i^2$.

The shape of the curve s^2 with some point from domains I–III as our initial point is shown in Fig. 3b.

Thus, if the initial point chosen in accordance with the above assumptions lies on the curves S_*, S_*°, then the curves s^1, s^2 are closed. For all other positions of the initial point the curves s^1, s^2 have S_* as their only limiting cycle.

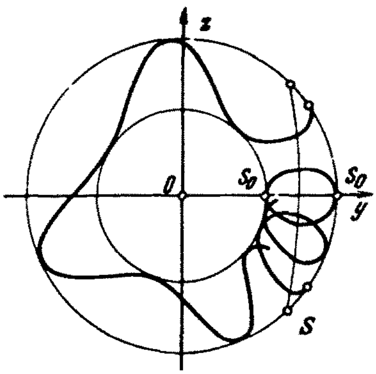

Fig. 4

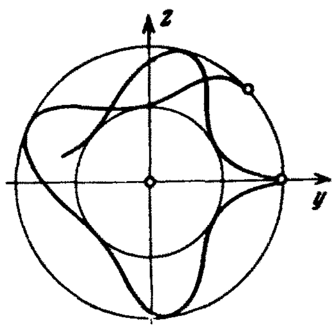

Fig. 5

b) In this case the curve s does not have a limiting cycle (Fig. 3c). The variable point M of the curve s as $\tau \to \infty$ executes an infinite number of revolutions about the origin. The curve s densely fills the whole of the ring G. The piece of the curve s which intersects the line L is of the type F_2; the remaining piece is of the type F_1.

c) Under this condition there exists a single closed trajectory S_* symmetric with respect to the y-axis which passes through all the points of intersection of the y-axis and the circles $\rho = \rho_1$, $\rho = \rho_2$. For all points of the domain G_2 except the points of the curve S_* we have $S_{i+1}^2 > S_i^2$; hence the angle φ receives a negative increment in the period $\tau = T$. If the initial point in the domain G_2 is chosen in such a way that $\varphi_0 > 0$, then the variable point M of the curve s^2 leaves the domain G_2, passes through the entire domain G_1, and re-enters G_2.

Fig. 4 shows the trajectory S_* and the curves s^2 for two different positions of the initial point.

The curves s in this case have S_* as their only limiting cycle.

We note that in each of variants (a), (b), (c) there exist two trajectories with cusps.

4. Let us take any point of the domain G other than the singular point M_2 of Eq. (1.9) as our initial point. For all points of the ring G other than the point M_2 we have $k - \rho^3 \cos\varphi > 0$, so that the angle φ always increases. The curve s (Fig. 5) is of the type F_1. The variable point M of the curve s executes an infinite number of revolutions about the origin as $\tau \to \infty$. The curve s densely fills the whole of the ring G. The singular point M_2 is the cusp of the curve s.

5. This case differs from the previous one only in the fact that the curve s does not have a cusp.

Thus the case $c > 0$, $k > 0$ has been covered in its entirety. When c has the opposite sign we obtain a curve s symmetric to that considered with respect to the y-axis; the case $c > 0$, $k < 0$ corresponds to a curve s symmetric to that studied with respect to the z-axis.

We note that system (1.8) is satisfied by the solutions $\rho = \rho_1$, $\rho = \rho_2$. The curve in this case is a circle, and the moving hodograph an ellipse.

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ON THE DYNAMICS OF A GAS BUBBLE IN A VISCOUS INCOMPRESSIBLE LIQUID

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It is generally agreed that intense oscillations of cavity bubbles without collapse, constitute one of the main reasons for the cavity erosion of materials. When the dimension of a cavity bubble reaches a certain limiting value, strong pressure pulses may occur in the surrounding liquid, which can cause erosion by local cyclic loads [1 and 2].

Oscillations of cavity bubbles in a viscous liquid, exhibit a number of distinctive features caused by the viscosity. Authors of [3 and 4] noted the fundamental influence of viscosity while investigating the behavior of a spherical cavity in a viscous, incompessible liquid. The existence of two different types of motion was discovered: bubbles which are smaller than a critical size, are filled slowly in an infinitely long time; the filling of large bubbles takes place rapidly with an unlimited accumulation of energy during collapse.

Below we find that, when the bubble in a viscous incompressible liquid is filled with gas, then two modes of motion exist, depending on the initial radius of the bubble, oscillatory or monotonically aperiodic.

Authors of [5] use dimensional analysis to derive a qualitative formula defining the critical bubble size D_* separating the inertial and inertialess mode of expansion of a gaseous sphere in a viscous liquid

$$D_* = \left(\frac{\mu\tau_*}{\rho}\right)^{1/2}$$

where μ and ρ are the dynamic viscosity and density of the liquid, respectively, and τ_* is the characteristic time of the process, determined experimentally. Below we derive a formula for the critical diameter of the gas bubble.

Let us suppose that a spherical gas bubble is situated in an infinite, viscous, incompressible liquid. We assume that the pressure and density of the gas are uniform throughout the bubble. This of course is true, provided that the velocity of the boundary of the gaseous sphere is much smaller than the velocity of sound in the gas at a given temperature. Viscosity of gas is assumed to be negligible. The following nonlinear, second order, differential equation [5] describes the variation in the radius of the bubble

$$R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 + \frac{4\mu}{\rho R}\frac{dR}{dt} + \frac{2\sigma}{\rho R} + \frac{p_0 - p'}{\rho} = 0 \tag{1}$$

and the initial conditions are

$$R = R_0, \quad dR/dt = 0 \quad \text{when} \quad t = 0$$

Here $R = R(t)$ is the radius of the bubble, σ denotes the surface tension of the liquid, p_0 denotes constant pressure of the liquid away from the bubble and p' denotes the pressure of gas within the bubble.

Assuming that the process of expansion and contraction of gas within the bubble is